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ON SINGULAR PERTURBATIONS AND PARABOLIC BOUNDARY LAYERS

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SUMMARY

A singular perturbation problem involving parabolic boundary layers is investigated. The second approximation for the boundary layer is constructed and it is shown that this approximation cannot be obtained by the usual perturbation method.

1. *Introduction*

In this report we shall study the problem of approximating the function U(x, y; ε) satisfying the differential equation:

$$\epsilon\left\{\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right\}-\frac{\partial U}{\partial y}=0,\quad x\geq 0,\ y\geq 0,\ 0<\epsilon\ll 1. \tag{1.1a}$$

And the boundary conditions:

$$U(x,\ 0)=0 \tag{1.1b}$$

$$U(0,\ y)=\phi(y),\quad \phi(0)=0. \tag{1.1c}$$

This problem belongs to the class of singular perturbation problems. In chapter 2 we summarize the results of Eckhaus and De Jager [2], who applied the singular perturbation method to the above problem. In chapter 3 we approach the problem from another point of view, and solve (1.1) by means of Green's theorem, which yields the exact solution. In chapter 4 we construct a uniformly valid expansion of this solution with respect to the parameter ε. Two reasons justify our way of treating the problem. Firstly, the properties of the solution in a neighbourhood of the origin are investigated; this is necessary as is shown in 2.1. Secondly, in chapter 5 we approximate the solution of the same equation in a bounded region by means of the results obtained in the chapters 3 and 4.

2. *Solution by Singular Perturbation Method*

2.1 The Parabolic Boundary-Layer.

In this chapter we summarize some of the results of [2]. In accordance with the singular perturbation theory we introduce a local coordinate

$$\xi=\frac{x}{\sqrt{\epsilon}}. \tag{2.1}$$

From (1.1) we obtain as an approximation the parabolic boundary-layer-solution:

$$U_0(\xi, y) = \sqrt{\frac{2}{\pi}} \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2} \phi\left(y - \frac{\xi^2}{2t^2}\right) dt. \tag{2.2}$$

Let $U(x, y; \epsilon) = U_0\left(\frac{x}{\sqrt{\epsilon}}, y\right) + Z(x, y; \epsilon)$, which yields for Z:

$$L_\epsilon(Z) = \epsilon\left\{\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2}\right\} - \frac{\partial Z}{\partial y} = -\epsilon \frac{\partial^2 U_0}{\partial y^2}, \tag{2.3a}$$

$$Z(x, 0) = Z(0, y) = 0. \tag{2.3bc}$$

We wish to apply a theorem (proved in [2]), which states: if $L_\epsilon(Z) = 0(\epsilon^\alpha)$ and on the boundary $Z = 0(\epsilon^\beta)$, then $Z = 0(\epsilon^{\min(\alpha, \beta)})$ for $x \geq 0$, $y \geq 0$.

However, application of this theorem is impossible, because of the unboundedness of (2.3a) in $\xi = y = 0$, which can be demonstrated by substituting (2.2) in (2.3a):

$$L_\epsilon(Z) = -\epsilon\sqrt{\frac{2}{\pi}}\left\{\frac{\xi}{(2y)^{3/2}} . e^{-\frac{\xi^2}{4y}} . \phi'(0) + \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2} . \phi''\left(y - \frac{\xi^2}{2t^2}\right) dt\right\}. \tag{2.4}$$

2.2 The Regularized Parabolic Boundary-Layer.

In (2.4), only the first term possesses a singular behaviour in the origin; a boundary condition $\bar{\phi}(y)$ with $\bar{\phi}'(0) = 0$, would therefore remove the first term and, thus, the singularity. We define $\bar{\phi}(y) = \phi(y) - \bar{Z}(0, y; \epsilon)$, where

$$\bar{Z}(0, y; \epsilon) = y\phi'(0) \exp\left(\frac{-y}{\epsilon^\alpha}\right) = 0(\epsilon^\alpha), \text{ with } \alpha > 0 \text{ arbitrary}, \tag{2.5a}$$

$$\bar{Z}(x, 0; \epsilon) = 0. \tag{2.5b}$$

We then obtain $\bar{U}_0(\xi, y)$ (the regularized parabolic boundary-layer). Setting up the problem for $\bar{Z}$, we conclude that it is now possible to apply the theorem mentioned in 2.1, because

$$L_\epsilon(\bar{Z}) = 0(\epsilon) + 0(\epsilon^{1-\alpha}), \text{ hence } \bar{Z}(x, y; \epsilon) = 0(\epsilon^{\min(1, \alpha, 1-\alpha)}). \tag{2.5c}$$

Optimal choice of α makes the remainder term $0(\sqrt{\epsilon})$. Finally it can be proved that $U_0 = \bar{U}_0 + 0(\sqrt{\epsilon})$.

In chapter 4 we shall show that:

a. the accuracy $0(\sqrt{\epsilon})$ is not the best estimation of the remainder term.
b. it is impossible to calculate higher-order approximations by the singular perturbation method.

3. *Solution by Means of Green's Theorem*

We introduce the transformation:

$$U(x, y; \epsilon) = u(x, y; \epsilon). e^{\frac{y}{2\epsilon}} \tag{3.1}$$

and obtain the differential equation of Helmholtz:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{u}{4\epsilon^2} = 0, \text{ for } x \geq 0, \; y \geq 0, \tag{3.2a}$$

with boundary conditions:

$$u(x, 0) = 0, \; u(0, y) = \phi(y).e^{-\frac{y}{2\epsilon}}. \tag{3.2b}$$

For the problem (3.2) we determine the Green's function $v(\xi, \eta; x, y)$ in the ξ, μ-plane with (x, y) fixed. We obtain:

$$v(\xi, \eta) = K_0\left(\frac{r_1}{2\epsilon}\right) - K_0\left(\frac{r_2}{2\epsilon}\right) - K_0\left(\frac{r_3}{2\epsilon}\right) + K\left(\frac{r_4}{2\epsilon}\right), \tag{3.3}$$

where r_1, r_2, r_3 and r_4 are distances between (ξ, η) and (x, y), (-x, y), (x, -y) and (-x, -y), respectively. Using transformation (3.1) we get from (3.2) and (3.3) with Green's theorem:

$$U(x, y; \epsilon) = \frac{-1}{\pi}\int_0^\infty \phi(t)\exp\left(\frac{y-t}{2\epsilon}\right)\frac{\partial}{\partial x}\left\{K_0\left(\frac{\sqrt{x^2+(t-y)^2}}{2\epsilon}\right) - K_0\left(\frac{\sqrt{x^2+(t+y)^2}}{2\epsilon}\right)\right\}dt. \tag{3.4}$$

4. *Asymptotic Expansion of the Solution*

4.1 Introductory Remarks

In this chapter we shall expand (3.4) in ϵ by defining the local coordinates:

$$\xi = \frac{x}{\epsilon^\alpha}, \; \eta - \frac{y}{\epsilon^\beta}, \; \alpha \geq 0, \; \beta \geq 0,$$

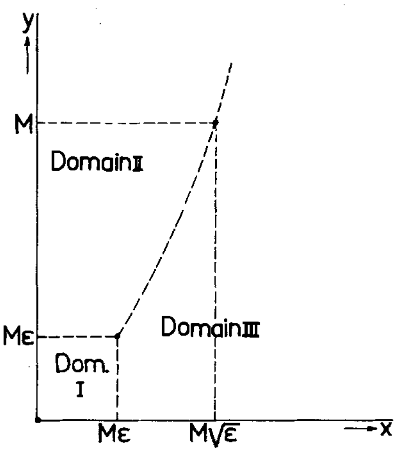
in which we express the solution. We define three sets of local coordinates in the following domains:

Domain I : $0 \leq x \leq M\epsilon$, $0 \leq y \leq M\epsilon$, M is an arbitrary large number independent of ϵ, see 4.2.

Domain II : $0 \leq x^2 \leq My\epsilon$, $M > y > M\epsilon$, see 4.3.

Domain III : $M \geq y > 0$, $x^2 > My\epsilon$, $x > \epsilon M$, see 4.4.

We remark that the technique of defining a set of local coordinates has been used in [3].



4.2. ϵ-Neighbourhood of the Origin.

We introduce the local coordinates and the integration variable: $\xi = \frac{x}{\epsilon}$, $\eta = \frac{y}{\epsilon}$, $p = \frac{t}{\epsilon}$. Substitution in (3.4) gives

$$U(\xi,\ \eta) = \frac{-1}{\pi}\int_0^\infty \phi(\epsilon p).\exp\left(\frac{\eta-p}{2}\right)\ \frac{\partial}{\partial\xi}\left\{K_0\left(\frac{r_1}{2}\right) - K_0\left(\frac{r_2}{2}\right)\right\}dp, \tag{4.1}$$

$$r = \sqrt{\xi^2 + (p-\eta)^2},\quad r_2 = \sqrt{\xi^2 + (p+\eta)^2}.$$

In this expression the parameter ϵ occurs only in $\phi(\epsilon p)$, so that, in general, one needs to solve the complete differential equation in an ϵ-neighbourhood of the origin. The order of magnitude of (4.1) depends on the behaviour of $\phi(y)$. If we suppose $\phi(y)$ to be analytic for $y \geq 0$, then (4.1) yields on asymptotic development:

$$U(\xi,\ \eta) = \sum_{n=1}^\infty v_n\,\epsilon^n,$$

$$v_n(\xi,\ \eta) = -\frac{\phi^{(n)}(0)}{\pi n!}\int_0^\infty p^n\ \exp\left(\frac{\eta-p}{2}\right)\ \frac{\partial}{\partial\xi}\left\{K_0\left(\frac{r_1}{2}\right) - K_0\left(\frac{r_2}{2}\right)\right\}dp. \tag{4.2}$$

Thus, in this case, the exact solution is $0(\epsilon)$ in an ϵ-neighbourhood of the origin.

4.3 The Parabolic Boundary-Layer.

We introduce in domain II:

$$\bar\xi = \frac{x}{\epsilon^{\frac{1}{2}(\beta+1)}},\quad \bar\eta = \frac{y}{\epsilon^\beta},\quad \text{for } \bar\xi \geq 0,\ \bar\eta \geq \delta > 0,\ 0 \leq \beta < 1, \tag{4.3}$$

with δ an arbitrary small positive number.

We separate (3.4) into two parts: U_a involving $K_0\left(\frac{\sqrt{x^2+(t-y)^2}}{2\epsilon}\right)$ and U_b involving $K_0\left(\frac{\sqrt{x^2+(t+y)^2}}{2\epsilon}\right)$.

4.3a. From (4.2) we obtain:

$$U_a = \sum_{n=1}^\infty V_{an}\,\epsilon^n,\quad V_{an}(\xi,\ \eta) = \frac{-\phi^{(n)}(0)}{\pi n!}\ \frac{\partial I_n}{\partial\xi}, \tag{4.4}$$

$$I_n(\xi, n) = \int_0^\infty w^n\ \exp\left(\frac{n-w}{2}\right)\ K_0\left(\frac{r_1}{2}\right)dw.$$

Substituting the integral representation:

$$K_0\left(\frac{r_1}{2}\right) = \int_1^\infty \frac{e^{-\frac{r_1 s}{2}}}{\sqrt{s^2-1}}\ ds$$

and defining new variables $p = w-\eta$, $t = r_1 s - p$, yields, after changing the order of integration: $I_n = I_{n1} + I_{n2}$,

$$I_{n1} = \int_0^{t_1} I_{Dn1}\ e^{-\frac{t}{2}}\ dt,\quad I_{n2} = \int_{t_1}^\infty I_{Dn2}\ e^{-\frac{t}{2}}\ dt,\quad t_1 = \eta + \sqrt{\eta^2 + \xi^2}.$$

$$I_{Dni} = \int_{p_i}^\infty \frac{(p+\eta)^n\ e^{-p}}{\sqrt{(t+p)^2 - r_1^2}}\ dp,\quad p_1 = -\frac{t}{2} + \frac{\xi^2}{2t},\quad p_2 = -\eta. \tag{4.5}$$

$$\frac{\partial I_n}{\partial \xi} = \left[I_{Dn1} - I_{Dn2} \right]_{t=t_1} . e^{-\frac{t_1}{2}} \frac{\partial t_1}{\partial \xi} + \int_0^{t_1} e^{-\frac{t}{2}} \frac{\partial I_{Dn1}}{\partial \xi} dt +$$

$$+ \int_{t_1}^{\infty} e^{-\frac{t}{2}} \frac{\partial I_{Dn2}}{\partial \xi} dt. \tag{4.6}$$

Suppose $U_a = \sum_{n=1}^{\infty} \left\{ U_{an1} + U_{an2} \right\}$, such that U_{an1} represents the contribution from the second term of (4.6) and U_{an2} from the third (the first term appears to be zero). In appendix 1 we prove that

$$U_{an2}(\bar{\xi}, \bar{\eta}) = 0(\epsilon^{n+1-\beta}). \tag{4.7}$$

An expression for $\frac{\partial I_{Dn1}}{\partial \xi}$ can be derived from (4.5) in an elementary way. We define a new integration variable $s = \xi^2/t$ and, moreover, we choose local coordinates (4.3), this gives

$$U_{an1}(\bar{\xi}, \bar{\eta}) = \frac{\phi^{(n)}(0)}{n! \sqrt{2\pi}} \int_{s_0}^{\infty} \frac{e^{-\frac{1}{2}s}}{\sqrt{s}} \left[\epsilon^{n\beta} \left(\bar{\eta} - \frac{\bar{\xi}^2}{2s} + \frac{s}{2} \epsilon^{1-\beta} \right)^n \right.$$

$$\left. - \sum_{m=0}^{n} \frac{a_{nm+1} \binom{n}{m}}{2^{n-m}} \epsilon^{n-m(1-\beta)} . \left(\bar{\eta} - \frac{\bar{\xi}^2}{2s} + \frac{s}{2} \epsilon^{1-\beta} \right)^m \right] ds,$$

$$a_{nn} = 1, \; a_{nm} = 1, \; 3, \; \dots \left\{ 2(n-m)-1 \right\}.$$

This expression can be expanded asymptotically with respect to ϵ (see appendix 2):

$$\sum_{n=1}^{\infty} U_{an1} = \sqrt{\frac{2}{\pi}} \int_{\frac{\bar{\xi}}{\sqrt{2\bar{\eta}}}}^{\infty} e^{-\frac{1}{2}t^2} \phi\left(\epsilon^{\beta} \bar{\eta} - \frac{\bar{\xi}^2 \epsilon^{\beta}}{2t^2} \right) dt$$

$$+ \epsilon \left[\phi'(0) \sqrt{\frac{2}{\pi}} \epsilon^{\beta-1} \int_{t_0}^{\frac{\xi}{\sqrt{2\bar{\eta}}}} e^{-\frac{1}{2}t^2} \left(\bar{\eta} - \frac{\bar{\xi}^2}{2t^2} \right) dt \right.$$

$$\left. + \frac{1}{\sqrt{2\pi}} \int_{\frac{\bar{\xi}}{\sqrt{2\bar{\eta}}}}^{\infty} e^{-\frac{1}{2}t^2} (t^2 - 1) . \phi'\left(\epsilon^{\beta} \bar{\eta} - \frac{\epsilon^{\beta} \bar{\xi}^2}{2t^2} \right) dt \right] + 0(\epsilon^2), \; t_0 = \sqrt{s_0}. \tag{4.8}$$

4.3b. The contribution from U_b is, because $K_1(p) = -K_0'(p)$:

$$U_b(x, y) = \frac{x}{2\pi\epsilon} \int_0^{\infty} \phi(t) \frac{K_1\left(\frac{\sqrt{x^2 + (t+y)^2}}{2\epsilon} \right)}{\sqrt{x^2 + (t+y)^2}} . \exp\left(\frac{y-t}{2\epsilon} \right) dt. \tag{4.9}$$

In the coordinates (4.3) the following estimation for the argument of K_1 holds:

$$\frac{1}{2\epsilon}\sqrt{\bar{\xi}^2\epsilon^{\beta+1} + (t+\bar{\eta}\epsilon^{\beta})^2} \geq \tfrac{1}{2}\delta\epsilon^{\beta-1} \gg 1.$$

The K_1-function can be expanded with respect to its large argument:
$K_1(p) = \sqrt{\frac{\pi}{2p}} \cdot e^{-p} \cdot \left\{1 + (p^{-1})\right\}$ for $p \gg 1$.

Applying this to (4.9), we obtain after repeated partial integration:

$$U_b = \sum_{n=1}^{\infty} U_{bn}, \quad U_{bn}(\bar{\xi}, \bar{\eta}) = \frac{\phi^{(n)}(0)\ \bar{\xi}\ \epsilon^{n+1-\beta}}{2n!\ \bar{\eta}\sqrt{\pi\bar{\eta}}} \exp\left(-\frac{\bar{\xi}^2}{4\bar{\eta}}\right) \cdot \left\{1 + 0(\epsilon^{1-\beta})\right\}, \quad (4.10)$$

$$U_{bn}(\bar{\xi}, \bar{\eta}) = 0(\epsilon^{n+1-\beta}) \text{ for } \bar{\eta} \geq \delta > 0. \quad (4.10a)$$

4.3c. The following expansion for (3.4) in domain II has been constructed:

$$U(\bar{\xi}, \bar{\eta}) = \sum_{n=1}^{\infty}\left\{U_{an1} + U_{an2} - U_{bn}\right\} \quad (4.11)$$

with U_{an1}, U_{an2} and U_{bn} as in (4.8), (4.7) and (4.10).

4.4. The Remaining Domain.

In domain III are defined the local coordinates

$$\hat{\xi} = \frac{x}{\epsilon^{\alpha}}, \quad \hat{\eta} = \frac{y}{\epsilon^{\beta}}, \quad \hat{\xi} \geq \delta > 0, \quad 0 \leq \alpha < \tfrac{1}{2}(\beta+1) < 1. \quad (4.12)$$

In this case the arguments of both Bessel functions are large, so that we can treat $U(\hat{\xi}, \hat{\eta})$ in the same way as U_b in 4.3b.
The result is:

$$U(\hat{\xi}, \hat{\eta}) = 0(\epsilon^N), \text{ with N an arbitrary large number.} \quad (4.13)$$

4.5. Uniformly Valid Expansion.

We wish to determine from the three local expansions one expansion, which holds in the complete region: $x \geq 0$, $y \geq 0$. This expansion has to be asymptotically equivalent to the local solutions in the corresponding domains. The term $U_1(x, y; \epsilon) = v_1\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right).\epsilon$ represents the solution (3.4) in the case that $\phi(y) = \phi'(0)y$. Applying the computations of section 4.3 to this solution, we obtain (4.11) for $n = 1$, in which

$$U_{a11}(\bar{\xi}, \bar{\eta}) = U_{10}(\bar{\xi}, \bar{\eta}) + \epsilon U_{11}(\bar{\xi}, \bar{\eta}) + 0(\epsilon^2)$$

in accordance with (4.8). Therefore, in

$$U_1(x, y; \epsilon) = U_{10}\left(\frac{x}{\sqrt{\epsilon}}, y\right) + \epsilon\left\{v_1\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right) - \frac{1}{\epsilon}U_{10}\left(\frac{x}{\sqrt{\epsilon}}, y\right)\right\}$$

the coefficient of ϵ is $0(1)$ in the domains I and II, in domain III (4.13) is satisfied. When we continue for $\phi(y) = \phi^{(n)}(0).y^n/n!$, $n = 2, 3, \ldots,$ and sum these expansions for U_n, we get the uniformly valid expansion for (3.4). The first two terms are:

$$U(x, y; \epsilon) = U_0\left(\frac{x}{\sqrt{\epsilon}}, y\right) + \epsilon U_1(x, y; \epsilon) + 0(\epsilon^2), \quad (4.14)$$

$$U_0\left(\frac{x}{\sqrt{\epsilon}},\ y\right) = \sqrt{\frac{2}{\pi}} \int_{\frac{x}{\sqrt{2y\epsilon}}}^{\infty} e^{-\frac{1}{2}t^2} \cdot \phi\left(y - \frac{x^2}{2\epsilon t^2}\right)dt, \tag{4.14a}$$

$$U_1(x,\ y;\ \epsilon) = v_1\left(\frac{x}{\epsilon},\ \frac{y}{\epsilon}\right) - \frac{\phi'(0)}{\epsilon}\sqrt{\frac{2}{\pi}} \int_{\frac{x}{\sqrt{2y\epsilon}}}^{\infty} e^{-\frac{1}{2}t^2} \cdot \left(y - \frac{x^2}{2\epsilon t^2}\right)dt +$$

$$+ \tfrac{1}{2}\sqrt{\frac{2}{\pi}} \int_{\frac{x}{\sqrt{2y\epsilon}}}^{\infty} e^{-\frac{1}{2}t^2} \cdot (t^2 - 1). \left\{\phi'\left(y - \frac{x^2}{2\epsilon t^2}\right) - \phi'(0)\right\}dt. \tag{4.14b}$$

5. *Application to a Bounded Region.*

The results of section 4.5 can be used to construct an asymptotic approximation of (1.1a) with boundary conditions in a bounded region. In this case we do not need the exact solution of the problem, which is, moreover, practically impossible to obtain.

$$\epsilon\left\{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right\} - \frac{\partial V}{\partial y} = 0,\ 0 \le x \le 1,\ 0 \le y \le 1. \tag{5.1}$$

$$V(x,\ 0) = f_1(x), \tag{5.1a}$$

$$V(x,\ 1) = f_2(x), \tag{5.1b}$$

$$V(0,\ y) = g_1(y),\ f_1(0) = g_1(0),\ f_2(0) = g_1(1) \tag{5.1c}$$

$$V(1,\ y) = g_2(y),\ f_1(1) = g_2(0),\ f_2(1) = g_2(1). \tag{5.1d}$$

For the first two terms of the approximation we obtain (as in [2]):

$$V = V_0 + \epsilon V_1 + 0(\epsilon^2), \tag{5.2}$$

$$V_0(x,\ y;\ \epsilon) = W_0(x,\ y) + U_{A0}\left(\frac{x}{\sqrt{\epsilon}},\ y\right) + U_{B0}\left(\frac{1-x}{\sqrt{\epsilon}},\ y\right) + T_0(x,\ y;\ \epsilon) \tag{5.2a}$$

$$V_1(x,\ y;\ \epsilon) = W_1(x,\ y) + U_{A1}(x,\ y;\ \epsilon) + U_{B1}(1-x,\ y;\ \epsilon). \tag{5.2b}$$

$W = W_0 + \epsilon W_1$ expresses the part of the solution, which satisfies (5.1a) by $W_0(x,\ y) = f_1(x)$, $W_1(x,\ y) = yf_1''(x)$.

$U_A = U_{A0} + \epsilon U_{A1} + 0(\epsilon^2)$ is the part which satisfies $U_A(0,\ y) = g_1(y) - W(0,\ y)$, it has the form (4.14). Likewise for $U_b(1-x,\ y)$ with $U_B(1,\ y) = g_2(y) - W(1,\ y)$.

Finally for $T(x,\ y)$, satisfying the remaining boundary $T(1,\ y) = f_2(x) - W(x,\ 1)$, we have a singular perturbation problem. It is solved by introducing the local coordinate $\eta = (1-y)/\epsilon$. The solution of the reduced equation is $T_0(x,\ y;\ \epsilon) = T(1,\ y)\exp((y-1)/\epsilon)$.

(5.2) is valid in the bounded region, excepted arbitrary small neighbourhoods of the points (0.1) and (1.1).

6. *Conclusions.*

We summarize the results of chapter 4. and 5. Considering the differential equation:

$$\epsilon\left\{\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right\}-\frac{\partial U}{\partial y}=0,\ x\geq 0,\ y\geq 0,\ 0<\epsilon\ll 1,$$

with boundary conditions $U(x,0)=0$, $U(0,y)=\phi(y)$, $\phi(0)=0$, we conclude:

a. In an ϵ-neighbourhood of the origin the exact solution has to be determined. The order of magnitude depends on the behaviour of $\phi(y)$.
b. When $\phi(y)$ is analytic for $y\geq 0$, then in an ϵ-neighbourhood of the origin the exact solution is $0(\epsilon)$ and can be expanded as (4.2).
c. The parabolic boundary-layer (4.14a) turns out to be a uniformly valid approximation of $U(x, y; \epsilon)$ with a remainder term $0(\epsilon)$
d. A uniformly valid approximation of $U(x, y; \epsilon)$ has been obtained with an accuracy $0(\epsilon^2)$. This approximation cannot be constructed by the usual (iterative) method, as it contains the exact solution of the differential equation with the reduced boundary condition $\phi(y)=\phi'(0)y$.

Finally we have established a method for finding higher-order approximations for the problem in a bounded region.

APPENDIX 1.

The contribution of A_{an2} to the solution is estimated as follows:

$$\frac{\partial I_{n2}}{\partial \xi}=\xi\int_{t_1}^{\infty}\int_{-\eta}^{\infty}\frac{(p+\eta)^n e^{-p-\frac{1}{2}t}}{(2tp+t^2-\xi^2)^{3/2}}\,dp\,dt-\left[I_{Dn2}\right]_{t=t_1}.e^{-\frac{t_1}{2}}\frac{\partial t_1}{\partial \xi}$$

The second term vanishes in (4.6). Introducing new integration variables $r=p+\eta$, $s=t-2\eta$, yields in the coordinates of (4.3):

$$U_{an2}(\bar{\xi},\bar{\eta})=\frac{-\phi^{(n)}(0)\,\bar{\xi}}{\pi\,n!}\epsilon^{n+1-\beta}\int_{s_1}^{\infty}\int_0^{\infty}\frac{r^{n-\frac{3}{2}}e^{-r-\frac{1}{2}s}drds}{\left\{4\bar{\eta}+2s\epsilon^{1-\beta}+\frac{s^2\epsilon^{1-\beta}+2s\bar{\eta}-\bar{\xi}^2}{r}\right\}^{3/2}}$$

We brought U_{an2} in a form, such that estimation is easy:
$U_{an2}(\bar{\xi},\bar{\eta})=0(\epsilon^{n+1-\beta})$ for $\bar{\eta}\geq\delta>0$.

APPENDIX 2.

Determining the first two terms of the expansion of $\sum_{n=1}^{\infty}U_{an1}$, we define

$$\sum_{n=1}^{\infty}U_{an1}=\sum_{i=1}^{3}U_{Ai},\quad U_{Ai}=\sum_{n=1}^{\infty}\frac{\phi^{(n)}(0)}{n!\sqrt{2\pi}}\int_{s_0}^{\infty}\frac{e^{-\frac{1}{2}s}}{\sqrt{s}}I_i\,ds,\quad I_1=\epsilon^{n\beta}\left(\bar{\eta}-\frac{\bar{\xi}^2}{2s}\right)^n,$$

$$I_2=\sum_{m=1}^{n=1}\frac{\binom{n}{m}}{2^m}\epsilon^{m+(n-m)\beta}.\left(\bar{\eta}-\frac{\bar{\xi}^2}{2s}\right)^{n-m}.s^m,$$

$$I_3 = \sum_{m=0}^{n=1} \frac{-a_{n,m+1}\binom{n}{m}}{2^{n-m}} \epsilon^{n-m(1-\beta)} \left(\bar{\eta} - \frac{\bar{\xi}^2}{2s} + \frac{s\epsilon^{1-\beta}}{2}\right)^m .$$

The term U_{A1}:

Define $t = \dfrac{\bar{\eta} + \bar{\eta}^2 + \bar{\xi}^2\epsilon^{1-\beta}}{2\bar{\eta}} \cdot s$

or: $s = \dfrac{t}{\left(1+\dfrac{\bar{\xi}}{4\bar{\eta}^2}\epsilon^{1-\beta}\dots\right)} = t\left(1-\dfrac{\bar{\xi}}{4\bar{\eta}^2}\epsilon^{1-\beta}\dots\right).$

$$U_{A1} = \sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)}{n!\sqrt{2\pi}} \epsilon^{n\beta} \int_{\frac{\bar{\xi}^2}{2\bar{\eta}}}^{\infty} \frac{e^{-\frac{1}{2}t}}{\sqrt{t}} \left[\bar{\eta} - \frac{\bar{\xi}^2}{2t} - \frac{\bar{\xi}^2}{2t}\left(\frac{\bar{\xi}^2}{4\bar{\eta}^2}\epsilon^{1-\beta}\dots\right)\right]^n .$$

$$\cdot\left(1+\frac{t\bar{\xi}^2}{8\bar{\eta}^2}\epsilon^{1-\beta}\dots\right)\left(1-\frac{\bar{\xi}^2}{8\bar{\eta}^2}\epsilon^{1-\beta}\dots\right)\left(1-\frac{\bar{\xi}^2}{4\bar{\eta}^2}\epsilon^{1-\beta}\dots\right) dt.$$

The expression consists of terms $0(\epsilon^{n+k(1-\beta)})$ with $n = 1, 2, 3, \dots$ $k = 1, 2, 3, \dots$.

For $k = 0$, $n = 1, 2, 3, \dots$ they form the first term of the asymptotic expansion, which is $0(1)$.

For $k = 1$, $n = 2, 3, 4, \dots$ the terms are $0(\epsilon)$.

For $n = 1$, $k = 1, 2, 3, \dots$ they are also $0(\epsilon)$, however, this set is not summed. We add this contribution to the solution by integrating over the original integration variable $s_0 \leq s \leq \bar{\xi}^2/2\bar{\eta}$ with $n = 1$.

For $k = 2, 3, 4, \dots$ $n = 2, 3, 4, \dots$ the terms are $0(\epsilon^2)$, so that

$$U_{A1} = \sum_{n=1}^{\infty} \frac{^{(n)}(0)}{n!\sqrt{2}} \cdot\epsilon^{n\beta}\cdot \int_{\frac{\bar{\xi}}{2\bar{\eta}}}^{\infty} \frac{e^{-\frac{1}{2}t}}{\sqrt{t}}\left(\bar{\eta} - \frac{\bar{\xi}^2}{2t}\right)^n dt +$$

$$\epsilon\left[\frac{\phi'(0)}{\sqrt{2\pi}}\epsilon^{\beta-1}\int_{s_0}^{\frac{\bar{\xi}^2}{2\bar{\eta}}} \frac{e^{-\frac{1}{2}s}}{\sqrt{s}}\left(\bar{\eta} - \frac{\bar{\xi}^2}{2s}\right) ds +\right.$$

$$\left.\sum_{n=2}^{\infty} \frac{\phi^{(n)}(0)}{n!\sqrt{2\pi}}\epsilon^{(n-1)\beta}\int_{\frac{\bar{\xi}^2}{2\bar{\eta}}}^{\infty} \frac{e^{-\frac{1}{2}t}}{\sqrt{t}}\left\{\left(\bar{\eta} - \frac{\bar{\xi}^2}{2t}\right)^n (t-1)\frac{\bar{\xi}^2}{8\eta^2} - n\left(\bar{\eta} - \frac{\bar{\xi}^2}{2t}\right)^{n-1}\cdot\frac{\bar{\xi}^4}{8t\bar{\eta}^2}\right\} dt\right] + 0(\epsilon^2).$$

Taking the Taylor-series together, we obtain, after a simplification by partial integration, and after a same computation for U_{Ai}, $i = 2, 3$:

$$\sum_{n=1}^{\infty} U_{an1} = \frac{1}{2\pi}\int_{\frac{\bar{\xi}^2}{2\bar{\eta}}}^{\infty} \frac{e^{-\frac{1}{2}s}}{\sqrt{s}}\phi\left(\epsilon\bar{\eta} - \frac{\bar{\xi}^2\epsilon^{\beta}}{2s}\right) ds +$$

$$\epsilon\left[\frac{\phi'(0)}{\sqrt{2\pi}}\epsilon^{\beta-1}\int_{s_0}^{\frac{\bar{\xi}^2}{2\bar{\eta}}} \frac{e^{-\frac{1}{2}s}}{\sqrt{s}}\left(\bar{\eta} - \frac{\bar{\xi}^2}{2s}\right) ds + \frac{1}{2\sqrt{2\pi}}\int_{\frac{\bar{\xi}^2}{2\bar{\eta}}}^{\infty} \frac{e^{-\frac{1}{2}s}}{\sqrt{s}}(s-1)\,\phi'\left(\epsilon^{\beta}\bar{\eta} - \frac{\epsilon^{\beta}\bar{\xi}^2}{2s}\right) ds\right] + 0(\ ^2).$$

This result has been used in section 4.3 with $t = \sqrt{s}$.

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REFERENCES

1. A.W. Vasileva — Non-linear differential equations containing a small parameter multiplying the highest derivative, Russ. Math. Surv., vol. 18, 1963, p. 13-84.

2. W. Eckhaus and E.M. de Jager — Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type, Arch. for Rat. Mech. and An., vol. 23, 1966, p. 26-86.

3. S. Kaplun and P.A. Lagerström — Asymptotic expansions of Navier-Stokes solutions for small Re-numbers, J. of Math. and Mech. vol. 6, 1957, p. 585.

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